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LETTER TO THE EDITOR

Eigenvalue correlations on hyperelliptic Riemann surfaces

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Abstract
In this letter we compute the functional derivative of the induced charge density, on a thin conductor, consisting of the union of $g+1$ disjoint intervals, $J := \cup_{j=1}^{g+1}(a_j, b_j)$, with respect to an external potential. In the context of random matrix theory this object gives the eigenvalue fluctuations of Hermitian random matrix ensembles where the eigenvalue density is supported on J.

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1. Introduction

Consider the minimization problem

$$E = \inf_{\mu \in \mathcal{A}} \left[-\int\int \log|x-t|\,\mu(x)\mu(t)\,\mathrm{d}x\,\mathrm{d}t + \int \mathsf{v}(x)\mu(x)\,\mathrm{d}x \right] \tag{1}$$

where the set $\mathcal{A}$ consists of all positive Lebesgue measures $\mu(t)\,\mathrm{d}t$ such that $\int \mu(t)\,\mathrm{d}t = 1$. The above formula describes the electrostatic equilibrium in which charges are placed on the real line in the presence of an external field $\mathsf{v}(x)$. For analytic external fields it is well known that the infimum (1) is attained at a unique measure $\sigma(x)\,\mathrm{d}x$ which is called the induced charge density. Moreover, the support of the induced charge density is generically characterized by a finite number of disjoint intervals $J := \cup_{j=1}^{g+1}(a_j, b_j)$ [11].

At electrostatic equilibrium, the induced charged density $\sigma(x)$, $x \in J$, satisfies the following integral equation

$$\mathsf{v}(x) - 2\int_J \ln|x-t|\,\sigma(t)\,\mathrm{d}t = \mathsf{A} = \text{constant} \qquad x \in J. \tag{2}$$

where the constant A is the Lagrange multiplier which fixes the constraint

$$\int_J \sigma(x)\,\mathrm{d}x = 1. \tag{3}$$

To determine σ, we convert (2) into a singular integral equation by taking the derivative of (2) w.r.t. x; that is,

$$2P\int_J \frac{\sigma(t)\,\mathrm{d}t}{x-t} = \frac{\mathrm{d}\mathsf{v}(x)}{\mathrm{d}x} \qquad x \in J \tag{4}$$

where P denotes the principal value of the singular integral. For generic potential v such that $\mathsf{v}'(x)$ is Hölder continuous[1], the solution, σ, of the singular integral equation (4), which is bounded at the endpoints of J, is necessarily zero there [10]; that is,

$$\sigma(a_j) = 0 = \sigma(b_j) \qquad j = 1, \ldots, g+1.$$

The endpoints $\{a_j, b_j\}_{j=1}^{g+1}$ of the support of σ are determined by (3),

$$\int_{b_j}^{a_{j+1}} \sigma(x)\,\mathrm{d}x = 0 \qquad j = 1, 2, \ldots, g \tag{5}$$

and by the moment conditions

$$\int_J \frac{x^k \mathsf{v}'(x)\,\mathrm{d}x}{\sqrt{\prod_{j=1}^{g+1}(x-a_j)(x-b_j)}} = 0 \qquad k = 0, \ldots, g \tag{6}$$

(see [9].) Equation (2) also arises from a mean-field approach to random Hermitian matrix ensembles [4]; $\sigma(x)$ is the averaged eigenvalue density, $\langle\varrho(x)\rangle$, where $\varrho(x) := \sum_{\nu=1}^{N}\delta(x - x_\nu)/N$ is the microscopic density of the eigenvalues, $\{x_\nu\}_{\nu=1}^{N}$, of a $N \times N$ Hermitian random matrix. The validity of the mean-field approximation, for large N, is discussed in [5]. An easy calculation shows that a functional derivative [3] of σ w.r.t. v is the density–density correlation function[2]:

$$\mathsf{C}_{\varrho\varrho}(x,t) := \frac{\delta\sigma(x)}{\delta\mathsf{v}(t)} = \frac{\delta\langle\varrho(x)\rangle}{\delta\mathsf{v}(t)} = \langle\varrho(x)\rangle\langle\varrho(t)\rangle - \langle\varrho(x)\varrho(t)\rangle \qquad x, t \in J \tag{7}$$

and must satisfy the obvious sum rules, $\int_J \mathsf{C}_{\varrho\varrho}(x,t)\,\mathrm{d}t = 0 = \int_J C_{\varrho\varrho}(x,t)\,\mathrm{d}x$. Furthermore, from (7), $\mathsf{C}_{\varrho\varrho}(x,t) = \mathsf{C}_{\varrho\varrho}(t,x)$. In this work we explicitly determine $\mathsf{C}_{\varrho\varrho}(x,t)$ as a function of the endpoints of the interval $J := \cup_{j=1}^{g+1}(a_j, b_j)$. It turns out that the density–density correlation function can be identified with the Bergman kernel of a Riemann surface which is a two-sheeted covering of the complex plane.

2. Determination of the density–density correlation function

First, we establish some notation. We consider the hyperelliptic Riemann surface $\mathcal{S}_g$ of genus g, defined by the equation

$$\mathcal{S}_g := \left\{(y,z),\ z \in \mathbb{CP}^1,\ y^2 = \prod_{j=1}^{g+1}(z-a_j)(z-b_j)\right\}.$$

[1] A function $f(x)$ is Hölder continuous if

$$|f(x_1) - f(x_2)| < c|x_1 - x_2|^\delta$$

for all x_1, x_2 in the domain of $f(x)$, for a constant $c > 0$ and $0 < \delta \leqslant 1$.

[2] Taking a functional derivative of

$$\langle\varrho(x)\rangle := \frac{\int \exp[-(H[\varrho] + \int \mathsf{v}(x')\varrho(x')\,\mathrm{d}x')]\varrho(x)\,\mathrm{D}\varrho}{\int \exp[-(H[\varrho] + \int \mathsf{v}(x')\varrho(x')\,\mathrm{d}x')]\,\mathrm{D}\varrho}$$

w.r.t. $\mathsf{v}(t)$ gives (7).

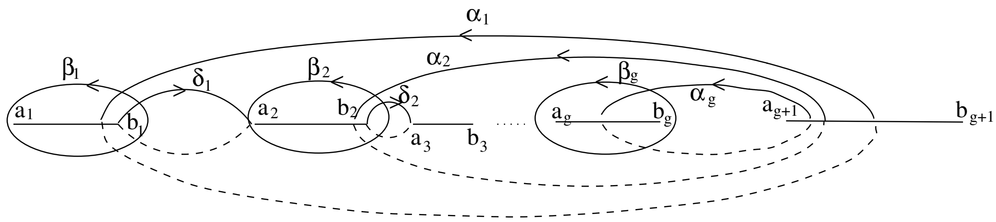

Figure 1. The cycles $\{\alpha_j, \beta_j, \delta_j\}_{j=1}^g$. The part of the cycles that lie in the lower sheet are indicated by broken lines.

The projection $(y, z) \to z$ defines $\mathcal{S}_g$ as a two-sheeted covering of the complex plane $\mathbb{C}$ cut along J. On $\mathcal{S}_g$ we define the canonical cycles $\{\alpha_k, \beta_k\}_{k=1}^g$ shown in figure 1. Let

$$\mathsf{U}_g(x) := \frac{\mathrm{i}}{\pi}\left(x^g + \sum_{j=0}^{g-1} \kappa_j x^j\right) \tag{8}$$

with κ_j, $j = 1, \ldots, g$, determined by

$$\int_{b_j}^{a_{j+1}} \frac{\mathsf{U}_g(x)}{y(x)}\,\mathrm{d}x = 0 \qquad j = 1, \ldots, g. \tag{9}$$

For completeness we first determine A as a function of v. Multiply (2) by $\mathsf{U}_g(x)/y(x)$, integrate w.r.t. x over J and noting that

$$\int_J \frac{\mathsf{U}_g(x)}{y(x)}\,\mathrm{d}x = 1$$

we find

$$\begin{aligned}\mathsf{A}[\mathsf{v}] &= -2\int_J \sigma(t)\,\mathrm{d}t \int_J \frac{\mathsf{U}_g(x)}{y(x)} \ln(b_{g+1} - x)\,\mathrm{d}x + \int_J \frac{\mathsf{v}(x)\mathsf{U}_g(x)}{y(x)}\,\mathrm{d}x \\ &= 2\mathsf{V}[J] + \int_J \frac{\mathsf{v}(x)\mathsf{U}_g(x)}{y(x)}\,\mathrm{d}x \end{aligned} \tag{10}$$

where

$$\mathsf{V}[J] := \int_{b_{g+1}}^{\infty} \left(\frac{\pi}{i}\frac{\mathsf{U}_g(t)}{y(t)} - \frac{1}{t}\right)\mathrm{d}t - \ln b_{g+1}. \tag{11}$$

The above integral is along any smooth path connecting b_{g+1} and ∞^+ laying on the upper half-plane of the upper sheet. So in the absence of the external field, $\mathsf{A}[0]/2$ is entirely determined by the endpoints of the conductor.

Note that

$$0 = \delta\left(\int_J \sigma(x)\,\mathrm{d}x\right) = \sum_{j=1}^{g+1}\left(\delta b_j \sigma(b_j) - \delta a_j \sigma(a_j)\right) + \int_J \delta\sigma(x)\,\mathrm{d}x = \int_J \delta\sigma(x)\,\mathrm{d}x. \tag{12}$$

Using similar calculations,

$$\int_{b_j}^{a_{j+1}} \delta\sigma(x)\,\mathrm{d}x = 0 \qquad j = 1, \ldots, g. \tag{13}$$

So performing a variation on (2) gives

$$\delta\mathsf{v}(x) - 2\sum_{j=1}^{g+1}\left(\ln|x - b_j|\,\sigma(b_j) - \ln|x - a_j|\,\sigma(a_j)\right) - 2\int_J \ln|x - t|\,\delta\sigma(t)\,\mathrm{d}t = \delta\mathsf{A}. \tag{14}$$

Also in this case, we multiply the above relation by $\mathsf{U}_g(x)/y(x)$, integrate w.r.t. x over J and, by (10), we obtain

$$\delta\mathsf{A} = \int_J \frac{\mathsf{U}_g(x)}{y(x)} \delta\mathsf{v}(x)\,\mathrm{d}x. \tag{15}$$

Taking a derivative w.r.t. x on (14) produces the singular integral equation

$$2P\int_J \frac{\delta\sigma(t)}{x-t}\,\mathrm{d}t = \frac{\mathrm{d}\delta\mathsf{v}(x)}{\mathrm{d}x} \qquad x\in J. \tag{16}$$

Now a possible form for $\delta\sigma(x)$, $x \in J$, reads

$$\delta\sigma(x) = \frac{1}{2\pi^2 y(x)} P\int_J \frac{y(t)}{t-x}\frac{\mathrm{d}\delta\mathsf{v}(t)}{\mathrm{d}t}\,\mathrm{d}t - \sum_{k=1}^{g}\frac{\varphi_k(x)}{2\pi^2 y(x)}\int_J \frac{\mathrm{d}\delta\mathsf{v}(t)}{\mathrm{d}t} y(t)\left(\int_{\alpha_k}\frac{\mathrm{d}s}{(t-s)y(s)}\right)\mathrm{d}t. \tag{17}$$

In order that $\delta\sigma$ actually satisfies (16), φ_k is taken to be a polynomial of degree $g-1$:

$$\varphi_k(x) = \sum_{l=1}^{g}\gamma_{kl}x^{g-l} \qquad k=1,\ldots,g \tag{18}$$

with yet undetermined γ_{kl} (see [6]). With this choice of φ_k, equations (12) and (16) are satisfied. Now $\varphi_k(x)\,\mathrm{d}x/y(x)$ is an Abelian differential of the first kind on the Riemann surface $\mathcal{S}_g$. If we choose γ_{kl} in such way that

$$\int_{\alpha_j}\frac{\varphi_k(x)}{y(x)}\,\mathrm{d}x = \delta_{kj} \tag{19}$$

then (13) is also satisfied. This completes the solution of (16). To see that (13) is satisfied, note that

$$\int_{b_j}^{a_{j+1}}\delta\sigma(x)\,\mathrm{d}x = -\frac{1}{2}\int_{\delta_j}\delta\sigma(x)\,\mathrm{d}x$$

where

$$\begin{aligned} \delta_j &= \alpha_j \setminus \alpha_{j+1} \qquad j=1,\ldots,g-1 \\ \delta_g &= \alpha_g. \end{aligned} \tag{20}$$

Here the cycles δ_j, $j=1,\ldots,g$, are shown in figure 1. Integrating by parts in (17) and noting that $y(a_j) = 0 = y(b_j)$ gives

$$\delta\sigma(x) = P\int_J \mathsf{C}_{\varrho\varrho}(x,t)\delta\mathsf{v}(t)\,\mathrm{d}t \qquad x\in J \tag{21}$$

where

$$\mathsf{C}_{\varrho\varrho}(x,t) := \frac{\partial}{\partial t}\left(\frac{y(t)}{2\pi^2 y(x)(x-t)} - \sum_{k=1}^{g}\frac{\varphi_k(x)y(t)}{2\pi^2 y(x)}\int_{\alpha_k}\frac{\mathrm{d}s}{(s-t)y(s)}\right) \qquad x,t\in J. \tag{22}$$

An inspection of (22) shows that sum rules are satisfied. Furthermore, $\mathsf{C}_{\varrho\varrho}(x,t)$ can be identified with the Bergman kernel of the Riemann surface $\mathcal{S}_g$ and is therefore symmetric under the exchange of x and t, namely $\mathsf{C}_{\varrho\varrho}(x,t) = \mathsf{C}_{\varrho\varrho}(t,x)$ [8, p 218].

A more direct way to check the symmetry property of $\mathsf{C}_{\varrho\varrho}(x,t)$ is shown below. Let $\pi_j(t)\,\mathrm{d}t/y(t)$ be an Abelian differential of the second kind with vanishing α periods

$$\int_{\alpha_k}\frac{\pi_j(t)}{y(t)}\,\mathrm{d}t = 0 \qquad j,k=1,\ldots,g \tag{23}$$

and with behaviour at infinity

$$\frac{\pi_j(t)}{y(t)}\,\mathrm{d}t \sim \pm\left(t^{j-1} + \mathrm{O}(t^{-2})\right)\mathrm{d}t \qquad t \sim \infty^{\pm}. \tag{24}$$

The quantity π_j is a polynomial in t of degree $g + j$:

$$\pi_j(t) = \Gamma_0 t^{g+j} + \Gamma_1 t^{g+j-1} + \ldots + \Gamma_j t^g + \mathsf{a}_1 t^{g-1} + \mathsf{a}_2 t^{g-2} + \ldots + \mathsf{a}_g \tag{25}$$

where the constants a_j's are determined by (23) and the constants Γ_k's determined by (24) are the coefficients of the expansion

$$y(z) \sim z^{g+1}\left(\Gamma_0 + \frac{\Gamma_1}{z} + \frac{\Gamma_2}{z^2} + \ldots\right) \qquad z \sim \infty^{+}. \tag{26}$$

From the Riemann bi-linear relations the second term in (22) can be expressed in terms of $\pi_j(t)$, without involving the constants γ_{jk} [1,7]. So,

$$2\pi^2 \mathsf{C}_{\varrho\varrho}(x,t) = \frac{y'(t)}{y(x)(x-t)} + \frac{y(t)}{y(x)(x-t)^2} + \frac{1}{y(x)y(t)}\sum_{k=1}^{g} x^{g-k}\sum_{j=1}^{k}(2j)\Gamma_{k-j}\pi_j(t). \tag{27}$$

The reader can now check that $\mathsf{C}_{\varrho\varrho}(x,t) - \mathsf{C}_{\varrho\varrho}(t,x)$ vanishes identically in x and t. For $g = 1$ this symmetry can be established in a straightforward calculation. The kernel $\mathsf{C}_{\varrho\varrho}$ will be used, in a later publication, for computing the distribution functions of linear statistics which is of interest in random matrix theory. For a discussion concerning linear statistics, see [2].

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